A CLASS OF EXACT SOLUTIONS OF THE EQUATIONS OF IDEAL PLASTICITY

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1. We consider the equations describing slow nonstationary plastic flows [1]:

$$\frac{\partial u_i}{\partial t} = \frac{\partial p}{\partial x_i} + \frac{\partial s_{ij}}{\partial x_j}, \quad s_{ij}s_{ij} + 2k_s^2, \tag{1.1}$$
$$2s_{ij} = \lambda\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0,$$

where u_1, u_2, u_3 are the components of the velocity vector; p is the hydrostatic pressure; s_{ij} are the components of the stress tensor deviator; λ is a nonnegative function; k_s is the flow limit for pure shear; repeated subscripts indicate summation; a subscript following a comma will indicate differentiation with respect to a spatial variable.

It should be remarked that the system of equations (1.1) is widely employed in engineering calculations.

2. It is well known that the system of equation (1.1), after the elimination from it of λ and s_{ij}, admits a group of continuous transformations [2] generated by the operators:

$$X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i} \quad (i = 1, 2, 3),$$
$$Z_1 = x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} + u_2\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_2},$$
$$T_1 = x_2\frac{\partial}{\partial u_3} - x_3\frac{\partial}{\partial u_2}, \quad S = \varphi(t)\frac{\partial}{\partial p},$$
$$L_i = g_i(t)\frac{\partial}{\partial u_i} - x_i g_i'(t)\frac{\partial}{\partial p} \quad (i = 1, 2, 3),$$
$$M = t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i}, \quad N = t\frac{\partial}{\partial t} + u_i\frac{\partial}{\partial u_i}.$$

An additional four operators, Z_2, Z_3 and T_2, T_3, may be obtained from Z_1, T_1 through a circular permutation of the subscripts.

The indicated group is infinite-dimensional since $g_i(t)$ and $\phi(t)$ are arbitrary functions from the class C^∞.

3. We consider an invariant solution of the system (1.1) on the subgroup $N = t\,\partial/\partial t + u_i\,\partial/\partial u_i$. We seek this solution in the form

$$u_1 = atx_1, \; u_2 = atx_2, \; p = cx_3 + p(x_1, x_2), \tag{3.1}$$
$$u_3 = -2atx_3 + 2\sqrt{6}atf(x_1, x_2).$$

The components of the stress tensor are then

$$s_{11} = s_{22} = \varkappa\lambda^*, \; s_{12} = 0, \; s_{33} = -2\varkappa\lambda^*, \tag{3.2}$$
$$s_{13} = \varkappa\lambda^* f_{,1}, \; s_{23} = \varkappa\lambda^* f_{,2}, \; \varkappa = \operatorname{sgn} a,$$
$$\lambda = \frac{\lambda^*}{|a|t} = \frac{k_s}{|a|t\sqrt{6}}(1 + (\nabla f)^2)^{-1/2}, \quad \nabla f = f_{,1} + f_{,2}.$$

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Here the function f is determined from the equation

$$2\sqrt{6}af = k_s\varkappa \operatorname{div}\frac{\nabla f}{\sqrt{1+(\nabla f)^2}} + c, \tag{3.3}$$

where c is an arbitrary constant and p can be found from the relations

$$ax_1 = \frac{\partial p}{\partial x_1} + \frac{\partial s_{11}}{\partial x_1}, \quad ax_2 = \frac{\partial p}{\partial x_2} + \frac{\partial s_{22}}{\partial x_2}. \tag{3.4}$$

From Eqs. (3.4) we obtain, taking the Eqs. (3.1) into account,

$$p = -\varkappa\lambda^* + \frac{1}{2}a\left(x_1^2 + x_2^2\right) + cx_3. \tag{3.5}$$

Consequently, to determine the velocity field (3.1) we need to solve Eq. (3.3) with appropriate boundary conditions.

4. We write Eq. (3.3) in the form

$$\operatorname{div}\frac{\nabla f}{\sqrt{1+(\nabla f)^2}} - bf = c_0, \tag{4.1}$$

where $b = (2\sqrt{6}|a|)/k_s$; $c = c_0\varkappa k_s$. We now use the solution (3.1) to describe plastic flows in a cylindrical passage whose generators are parallel to the x_3-axis and whose directrix is given by the equation $F(x_1, x_2) = 0$. We assume here that on the walls of the passage a uniformly distributed tangential stress is prescribed:

$$\tau_n = k_s \cos\alpha, \ 0 \leqslant \alpha \leqslant \pi/2,$$

to it there corresponds the boundary condition

$$\frac{\partial f}{\partial n}(1 + (\nabla f)^2)^{-1/2} = k_s \cos\alpha, \tag{4.2}$$

where n is the inner normal to the curve $F(x_1, x_2) = 0$. The problem (4.1), (4.2) arises in the study of equilibrium surfaces in the hydromechanics of weightlessness [3]. Moreover, if ℓ is a characteristic passage dimension, the quantity $b^2\ell$ is called the Bond number and defines the relationship between the gravitational and capillary forces.

5. In a three-dimensional problem one cannot expect, in the general case, to be able to solve the problem (4.1), (4.2) in analytical form. One can solve it through analytical approximations and numerical methods [3]. In particular, if b >> 1, we may seek a solution of the problem (4.1), (4.2) in the form of the asymptotic expansion

$$f_\varepsilon(x_1, x_2) \sim \sum_{i=1}^{\infty}\varepsilon^i\omega_i(x_1, x_2) + \varepsilon\sum_{i=0}^{\infty}\varepsilon^i v_i(h/\varepsilon, \varphi), \quad \varepsilon = \frac{1}{\sqrt{b}},$$

where ϕ is a parameter reckoned along the directrix F = 0 and h is the distance from a point to the directrix reckoned along the normal to F = 0. Here the functions ω_i are obtained from the condition of formally satisfying Eq. (4.1) and the functions v_i, of boundary layer type, compensate for the discrepancy in the boundary condition (4.2).

Remark. According to [3], to solve the problem (4.1), (4.2) one can also use variational methods, in particular, the method of local variations.

6. In the axially-symmetric case, if we consider the flow in a circular cylinder of radius r_0, we can make use of the method described in Section 5 for b >> 1. A second case, readily amenable to study, is the case $\nabla f \ll 1$. If this condition is satisfied, the Eq. (4.1), after linearization, is readily converted to

$$rf'' + f' - brf = rc. \tag{6.1}$$

A solution of Eq. (6.1), bounded for r = 0, has the form

$$f = c_0 I_0(r\sqrt{b}) - c/b,$$

where I_0 is the Bessel function of imaginary argument. The arbitrary constant c_0 is determined from the boundary condition (4.2), which, after linearization, may be written in the following way:

$$\left.\frac{\partial f}{\partial r}\right|_{r=r_0} = k_s \cos\alpha.$$

The constant c is determined from the condition requiring preservation of volume. For the case in question, the velocity field is

$$u_r = atr,\ u_\theta = 0,$$
$$u_z = -2atz + 2\sqrt{6}at(c_0 I_0(r\sqrt{b}) - c/b). \tag{6.2}$$

The components of the deformation rate tensor and the stress tensor are given by

$$e_r = at,\ e_\theta = at,\ e_z = -2at,$$
$$2e_{rz} = \sqrt{6}atf'_r,\quad e_{r\theta} = e_{\theta z} = 0,$$
$$\sigma_r = \sigma_\theta = -p + \frac{k_s}{\sqrt{3}},\quad \sigma_z = -p - \frac{2k_s}{\sqrt{3}},$$
$$\tau_{rz} = \sqrt{2}k_s f'_r,\ p = (1/2)ar^2 + cx_3 + c(t),\ \tau_{r\theta} = \tau_{z\theta} = 0.$$

We note that the condition $\nabla f \ll 1$ is realized, in particular, when $|\alpha - \pi/2| \ll 1$, i.e., when the friction on the walls of the passage is small.

Remark. The solution (6.2) can also be used in describing plastic flow on a contractible cylindrical pipe.

7. The planar problem is most readily amenable to analysis. In this case the Eq. (4.1) may be solved by quadrature and the resulting solution can be used to describe plastic flow between rigid plates, which draw together at a constant acceleration. For the planar case we write Eq. (4.1) in the form

$$\frac{d}{dx_1}\left(\frac{f_{,1}}{\sqrt{1+f_{,1}^2}}\right) - bf = c_0. \tag{7.1}$$

If in Eq. (7.1) we make the substitution $f_{,1} = z(f)$, we have

$$\frac{z'z}{(1+z^2)^{3/2}} = bf + c_0. \tag{7.2}$$

From Eq. (7.2) we have

$$f_{,1} = \sqrt{(b/2f^2 + c_0 f + c_1)^{-2} - 1}. \tag{7.3}$$

Integrating the latter for $\left|c_1 - \frac{c_0}{2b}\right| < 1$, we obtain

$$x_1(f) - x_1(0) = \frac{1}{\sqrt{b}}\left[E\left(\frac{\pi}{2}, k\right) - E(\varphi, k)\right] + \left(\frac{\sqrt{2}c_0}{4b} - \frac{A^2}{\sqrt{2}}\right)[K(k) - F(\varphi, k)],$$

where

$$A^2 = 1 + c_1 - \frac{c_0^2}{2b};\quad B^2 = 1 - c_1 + \frac{c_0^2}{2b};\quad \varphi = \arccos fB^{-1};\quad B = \sqrt{2}k.$$

But if $c_1 - c_0^2/2b < -1$, we then obtain from Eq. (7.3) the result

$$x_1(f) - x_1(B) = \frac{B\sqrt{2}}{b}\left(E\left(\frac{\pi}{2}, k\right) - E(\varphi, k)\right) - \frac{\sqrt{2}}{Bb}(K(k) - F(\varphi, k)),$$

where $\varphi = \arcsin\sqrt{\frac{B^2 - f^2}{B^2 - A^2}}$; $k = \frac{\sqrt{B^2 - A^2}}{B}$; $F(\phi, k)$ and $E(\phi, k)$ are elliptic integrals of the first and second kinds, respectively; $K(k) = F(\pi/2, k)$.

Consequently, in the planar case the velocity field is given by

$$u_1 = atx_1,\ u_2 = atx_2,\ u_3 = -2atx_3 + 2\sqrt{6}atf(x_1),$$

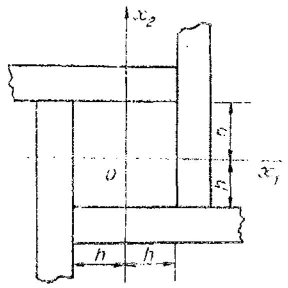

Fig. 1

and the stress tensor components are given by

$$\sigma_{11} = \sigma_{22} = (1/2)a\left(x_1^2 + x_2^2\right) + cx_3,$$

$$\sigma_{33} = \sqrt{\frac{2}{3}}\frac{k_s}{a}\left|\frac{b}{2}f^2 + c_0 f + c_1\right| + \sigma_{11},$$

$$\sigma_{12} = \sigma_{23} = 0,$$

$$\sigma_{13} = \frac{k_s}{a\sqrt{6}}\sqrt{1 - \left(\frac{b}{2}f^2 + c_0 f + c_1\right)^2}.$$

This solution can be used to describe the plastic flow of a beam in the shape of a parallelepiped of cross-sectional dimensions 2h by 2h and of length 2ℓ, compressed by four rigid plates, which move towards one another at constant acceleration. A cross section of the passage formed by the plates is shown in Fig. 1. The plates parallel to the Ox_1x_3-plane approach each other along the x_2-axis (smooth); the plates parallel to the Ox_2x_3-plane approach each other along the x_1-axis (rough). If we put $a = \omega/h$, then ω is the acceleration with which the plates approach each other along the x_1 and x_2 axes. The constants c_0 and c_1 are determined from the condition of incompressibility of the material and from the condition at the free end $x_3 = \ell$. This condition, just as in Prandtl's solution in [4], is satisfied in the sense of Saint-Venant. The plastic flow of a beam, compressed by four plates drawing together at specified rates, was considered in [5].

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